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THE RECONSTRUCTION OF ANALOG SIGNALS FROM THE SIGN OF THEIR NO1--ETC(U)  
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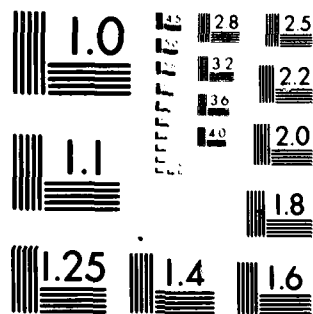
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THE RECONSTRUCTION OF ANALOG SIGNALS  
FROM THE SIGN OF THEIR NOISY SAMPLES

by

Elias Masry

ABSTRACT

The reconstruction of continuous-time signals  $s(t)$  from the sign of their (deliberately) contaminated samples is considered. Sequential, generally non-linear estimates of  $s(t)$  are established and their performance is studied; error bounds and convergence rates are derived. The signal  $s(t)$  need not be bandlimited. The convergence rates obtained here are faster than those obtained in [4] for nonsequential estimates. The degradation in the reconstruction of the signal, due to transmission over an arbitrary noisy channel, is also investigated and bounds on the additional error are obtained.

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## I. INTRODUCTION

This paper is concerned with the problem of reconstructing a continuous-time signal  $s(t)$ ,  $-\infty < t < \infty$ , from its sign (clipped version)  $\text{sgn}[s(t)]$ ,  $-\infty < t < \infty$ . In general the binary signal  $\text{sgn}[s(t)]$  does not uniquely determine  $s(t)$  even when  $s(t)$  is an analytic function. For example, when  $s(t)$  is a bandlimited function and the hard-limiter  $\text{sgn } x$  is replaced by a strictly monotonic transformation  $\psi(x)$ , then  $s(t)$  can be recovered from the bandlimited version of  $\psi[s(t)]$  by using a recursive algorithm based on the principle of contraction mapping; this was accomplished by Landau [1] for conventionally bandlimited functions and by Masry and Cambanis [2] for bandlimited (in the sense of Zakai) stochastic signals. No such results, however, are available when  $\psi(x) = \text{sgn } x$  whether  $s(t)$  is bandlimited or not (see, for example, [3]).

In [4] we proposed a digital scheme, modelled as a transmitter/receiver as in Figure 1, whereby the signal  $s(t)$  - not necessarily bandlimited - is sampled periodically at a fixed sampling rate  $W$ . The samples  $\{s(k/W)\}_k$  are deliberately contaminated by additive noise  $\{X_k\}_k$  having an appropriate distribution  $F(x)$  (in practical applications,  $\{X_k\}_k$  are computer generated random numbers drawn from the distribution  $F(x)$ ). The sign of the contaminated samples  $\{s(k/W) + X_k\}_k$  is then obtained, i.e.,  $\{Z_{W,k} = \text{sgn}[s(k/W) + X_k]\}_k$ . It was then shown in [4] that estimates  $\hat{s}_W(t)$  of  $s(t)$ , based on the  $\pm 1$  sequence  $\{Z_{W,k}\}_k$ , exist such that  $\hat{s}_W(t)$  converges (with probability one) to  $s(t)$  as the sampling rate  $W$  tends to infinity. Note that in the absence of  $\{X_k\}_k$ ,  $s(t)$  cannot be reconstructed from  $\{\text{sgn}[s(k/W)]\}_k$  as  $W \rightarrow \infty$ .

This paper continues the investigation begun in [4] and has several objectives: We first note that the estimates considered in [4] were nonsequential, i.e., they reconstruct  $\{s(t), t > 0\}$  from the entire data set  $\{Z_{W,k}\}_{k=0}^{\infty}$ . Our first objective is to establish sequential estimates of  $s(t)$  and study their

performance. We seek to obtain tight bounds on the moments of the error  $\hat{s}_W(t) - s(t)$  for a finite sampling rate  $W$ , the establishment of mean and probability one rates of convergence as  $W \rightarrow \infty$ , and a central limit theorem for the error  $\hat{s}_W(t) - s(t)$ . These results for the sequential estimates are sharper than those obtained in [4]; for example, the rate of convergence of the mean-square error is at most  $W^{-1/2}$  for the nonsequential estimates of [4] and is  $W^{-2/3}$  for the sequential estimates considered here. A second objective of this paper is the investigation of the effect of channel noise when the binary data  $\{Z_{W,k}\}_k$  is transmitted over a noisy channel (see Fig. 2). This is considered in Section III where bounds on the additional error in the estimation of  $s(t)$ , due to channel noise, are derived. Distinct results are obtained for the white and colored channel noise cases.

The approach used in [4] and in this paper, to deliberately contaminate the signal  $s(t)$  before quantization, can be viewed in two ways. One is that of "dithering" - a concept which has been used in the past for correlation function estimation [5] [6], digital match filtering [7], and other aspects of digital signal processing as reviewed in [8] (see also the paper by Root [9] in the context of communication through unspecified additive noise). Alternatively, it can be viewed as random quantization for deterministic continuous-time signals. The idea of random quantization has recently been advocated by Papantoni-Kazakos [10] for discrete-time stochastic signals and shown to be essential for stability under perturbations in the statistical description of the signals.

The organization of the paper is as follows. In Section II we consider the noise-free channel case and derive the convergence properties of the sequential estimate  $\hat{s}_W(t)$ . In Section III we consider the effect of channel noise on the performance of the estimate  $\hat{s}_W(t)$ . Both Sections II and III contain a discussion on the choice of the parameters of the transmitter/receiver

such that the reconstruction of the signal is achieved with an error not exceeding a prescribed level. Section IV is a collection of remarks on certain unresolved questions in this area. The Appendix contains certain auxiliary propositions needed for the derivations in Section II, as well as a supplement to Section III.

## II. THE RECONSTRUCTION OF THE SIGNAL (NOISE-FREE CHANNEL)

### A. Preliminaries

We consider the noise-free channel case, as depicted in Figure 1, and specify admissible distributions  $F(x)$  of  $\{X_k\}$ , linear systems  $L$ , and memoryless nonlinearities  $g(x)$ , such that  $\hat{s}_W(t)$  is a sequential estimate of  $s(t)$ ; the convergence properties of  $\hat{s}_W(t)$  are then obtained. Throughout this paper it is assumed that the signal  $s(t)$  belongs to the following class of signals.

Assumption A. Let  $b$  be a fixed known positive constant and let  $s(t)$ ,  $t \geq 0$ , be any uniformly continuous function satisfying  $|s(t)| \leq b$  for all  $t \geq 0$ .

The constant  $b$  is simply a peak constraint on the signal  $s(t)$ . Aside from the knowledge of  $b$ , the receiver structure and the convergence results of this paper are nonparametric in the signal. Next it is assumed that the contaminating random numbers  $\{X_k\}_{k=0}^{\infty}$  constitute a sequence of independent identically distributed random variables with a symmetric distribution  $F(x)$ . The following argument provides the rationale for the recovery scheme and determines the class of admissible distributions  $F(x)$ . Let  $X$  be a random variable with a symmetric distribution function  $F(x)$  and define the moment function  $\mu(s)$  by

$$\mu(s) = E[\text{sgn}(s + X)], \quad -\infty < s < \infty.$$

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$$\begin{aligned}
\mu(s) &= P[X \geq -s] - P[X < -s] \\
&= 1 - 2F(-s^-) \\
&= 2F(s) - 1, \quad -\infty < s < \infty.
\end{aligned}$$

Thus  $\mu(s)$  is strictly monotonic on an interval  $(-c, c)$  if and only if  $F(x)$  is strictly monotonic on  $(-c, c)$ . Any distribution  $F(x)$  which is strictly monotonic on an interval  $(-c, c)$  for some  $c \geq b$  is an admissible distribution for the sequence  $\{X_k\}$  in the transmitter. Now let

$$m(t) = E[\text{sgn}(s(t) + X)], \quad t \geq 0 \quad (2)$$

be the mean function of the hard-limiter output, with input  $s(t) + X$ , then it is seen that

$$m(t) = \mu[s(t)], \quad t \geq 0. \quad (3)$$

By the strict monotonicity of  $\mu(\cdot)$  over  $(-c, c)$ , for some  $c \geq b$ , we then have  $s(t) = \mu^{-1}(m(t))$ . Hence, in principle, an estimate  $\hat{s}(t)$  of  $s(t)$  can be obtained from an estimate  $\hat{m}(t)$  of  $m(t)$  via  $\hat{s}(t) = \mu^{-1}(\hat{m}(t))$ ;  $\hat{m}(t)$  can be obtained from the binary data  $\{Z_{W,k}\}_k$  in a linear recursive manner. This explains the structure of the recovery scheme in Fig. 1. Some refinement of the above argument is needed, however, since  $\hat{m}(t)$  need not take values in the interval  $[\mu(-b), \mu(b)]$  whereas  $m(t)$  does. This will become clear below.

Convergence results can be obtained for any admissible distribution  $F(x)$  specified above; however, in order to provide explicit bounds on the mean-square error of the estimates, we shall concentrate on three typical distributions. When  $X$  is uniform over  $[-b, b]$  we have

$$\mu_U(s) = \begin{cases} -1, & s < -b \\ s/b, & -b \leq s \leq b \\ 1, & b < s. \end{cases} \quad (4)$$

When  $X$  is normal  $N(0, \sigma^2)$  we have

$$\mu_N(s) = 2\Phi(s/\sigma) - 1, \quad -\infty < s < \infty, \quad (5)$$

where  $\phi(x)$  is the standard normal distribution function. When  $X$  is Laplacian ( $f(x) = (\alpha/2)\exp(-\alpha|x|)$ ) we have

$$\mu_L(s) = (\operatorname{sgn} s) (1 - e^{-\alpha|s|}), \quad -\infty < s < \infty. \quad (6)$$

We now specify the memoryless nonlinearity  $g(x)$  in the receiver by  $g(x) = \mu^{-1}(x)$  over an interval containing  $[\mu(-b), \mu(b)]$ , and by  $g(x) = 0$  elsewhere. For the three chosen distributions  $F(x)$  we have  $\mu_N(s)$  and  $\mu_L(s)$  are invertible over the entire real line while  $\mu_U(s)$  is invertible over  $[-b, b]$ , and we define  $g(x)$  as follows.

Assumption B. We say that (B) is satisfied if any one of (B1), (B2), or (B3) is satisfied.

(B1):  $X$  is uniform over  $[-b, b]$  and

$$g_U(x) = \begin{cases} bx & , \quad |x| \leq 1, \\ 0 & , \quad |x| > 1. \end{cases}$$

(B2):  $X$  is normal  $N(0, \sigma^2)$  and

$$g_N(x) = \begin{cases} \mu_N^{-1}(x) & , \quad |x| \leq \mu_N(c) \\ 0 & , \quad \text{otherwise} \end{cases}, \quad c = b + \epsilon, \quad \epsilon > 0.$$

(B3):  $X$  is Laplacian and

$$g_L(x) = \begin{cases} -\frac{1}{\alpha}(\operatorname{sgn} x) \ln(1 - |x|) & , \quad |x| \leq 1 - e^{-\alpha c} \\ 0 & , \quad \text{otherwise} \end{cases}, \quad c = b + \epsilon, \quad \epsilon > 0.$$

We next specify the linear system  $L$  in the receiver whose output  $\hat{m}_W(t)$  provides an estimate of the function  $m(t)$  given in (3).



We consider sequential estimates using a sliding window on blocks of a fixed size  $N$  of the data  $\{Z_{W,k}\}$ : First  $m(t)$  is estimated at the sampling points  $(k/W)_{k=0}^{\infty}$  by

$$\hat{m}_W(\frac{k}{W}) = \frac{1}{N} \sum_{i=0}^{N-1} Z_{W,i+k}, \quad k = 0, 1, \dots, \quad (7)$$

and then  $m(t)$  is estimated either by the step function

$$\hat{m}_W(t) = \sum_{k=0}^{\infty} \hat{m}_W(\frac{k}{W}) 1_{[\frac{k}{W}, \frac{k+1}{W})}(t) \quad (8a)$$

or by the piecewise linear function

$$\hat{m}_W(t) = \sum_{k=0}^{\infty} \{ (Wt-k) \hat{m}_W(\frac{k+1}{W}) + [1-(Wt-k)] \hat{m}_W(\frac{k}{W}) \} 1_{[\frac{k}{W}, \frac{k+1}{W})}(t). \quad (8b)$$

The estimate  $\hat{s}_W(t)$ ,  $t \geq 0$ , is then given by

$$\hat{s}_W(t) = g[\hat{m}_W(t)] \quad (9)$$

where  $g(x)$  is specified by Assumption (B) and  $\hat{m}_W(t)$  is given by (8a) or (8b).

Thus  $\hat{s}_W(t)$  is obtained sequentially with delay not exceeding  $N/W$  and  $(N+1)/W$  for (8a) and (8b), respectively. The estimate  $\hat{s}_W(t)$  determined by (8b) is continuous in  $t$  whereas the estimate determined by (8a) is a step function in  $t$ , and thus the former may be considered a more suitable estimate for the continuous signal  $s(t)$ . Note that when  $X$  is uniform over  $[-b, b]$ , we have

$$\hat{s}_W(t) = b \hat{m}_W(t), \quad t \geq 0 \quad (10)$$

since only the linear portion of  $g_U(x)$  is used (as by (8a) and (8b),  $|\hat{m}_W(t)| \leq 1$ ).

In this case, therefore, the estimate  $\hat{s}_W(t)$  is linear in the data  $\{Z_{W,k}\}_{k=0}^{\infty}$ .

#### B. Bounds on Mean-Square Error and Discussion

Our first result provides a bound on the mean-square error of the estimate  $\hat{s}_W(t)$ . It is stated in terms of the modulus of continuity  $\omega(s; \delta)$  of  $s(t)$ , which is defined for each  $\delta > 0$  by

$$\omega(s; \delta) = \sup_{\{t, t' \geq 0: |t-t'| < \delta\}} |s(t) - s(t')|,$$

and which tends to zero as  $\delta \rightarrow 0$  by the uniform continuity of  $s(t)$  over  $[0, \infty)$ . Eventhough we state all results for signals defined, and uniformly continuous, over the entire half line  $[0, \infty)$ , the same results are of course valid for continuous signals defined over finite intervals  $[0, T]$  with the modulus of continuity over  $[0, \infty)$  replaced by that over  $[0, T]$ .

Theorem 2.1. Under Assumptions (A), (B), the estimate  $\hat{s}_W(t)$  satisfies

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \omega^2(s; \frac{N}{\sqrt{3}W}) + K_2 \frac{1}{N}$$

uniformly in  $t \geq 0$ . The constants  $K_1$  and  $K_2$  are determined by (B) as follows:

$$\text{For (B1): } K_1 = 4, \quad K_2 = b^2$$

$$\text{For (B2): } K_1 = \frac{8}{\pi\sigma^2} K_2, \quad K_2 = \frac{1}{2\pi\sigma^2} e^{c^2/\sigma^2} (1+b^2/\epsilon^2)$$

$$\text{For (B3): } K_1 = 4\alpha^2 K_2, \quad K_2 = \alpha^{-2} e^{2\alpha c} (1+b^2/\epsilon^2)$$

Theorem 2.1 shows, in particular, that the estimate  $\hat{s}_W(t)$  converges to  $s(t)$  in the mean-square sense as  $W \rightarrow \infty$ , uniformly in  $t \geq 0$ , provided the block size  $N$  is chosen to depend on  $W$ ,  $N = N_W$ , such that

$$N_W \rightarrow \infty \text{ and } \frac{N_W}{W} \rightarrow 0 \text{ as } W \rightarrow \infty. \quad (11)$$

Proof. We first note that the function  $m(t) = \mu[s(t)]$  is also uniformly continuous on  $[0, \infty)$  since  $\mu(s)$ , given in (4)-(6), is differentiable over  $[-b, b]$  with

$$Q = \max_{|s| \leq b} \mu'(s) = \begin{cases} 1/b & , \text{ under (B1)} \\ \sqrt{2/\pi}/\sigma & , \text{ under (B2)} \\ \alpha & , \text{ under (B3),} \end{cases} \quad (12)$$

i.e.,

$$\omega(m; \delta) \leq Q \omega(s; \delta). \quad (13)$$

Next we obtain a bound on the mean-square error of the estimates  $\hat{m}_W(t)$ , given in (8a)-(8b). The corresponding bound for  $\hat{s}_W(t)$  follows then from Proposition 1 of the Appendix.

a) For the estimate (8a) and for each fixed  $t \geq 0$ , we have  $\hat{m}_W(t) = \hat{m}_W(k/W)$  where  $k$  is such that  $k/W \leq t < (k+1)/W$ . Since

$$E[Z_{W,k}] = u[s(k/W)] = m(k/W)$$

we have by (7)

$$E[\hat{m}_W(t)] = \frac{1}{N} \sum_{i=0}^{N-1} m\left(\frac{i+k}{W}\right)$$

which is a positive linear operator on the function  $m(u)$ ,  $u \geq 0$ , and by a well-known result in approximation theory [11, pp. 28-29]

$$|\text{Bias}[\hat{m}_W(t)]| = |E[\hat{m}_W(t)] - m(t)| \leq 2 \omega(m; \alpha_{W,k}(t))$$

where

$$\begin{aligned} \alpha_{W,k}^2(t) &= \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{i+k}{W} - t\right)^2 = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{i}{W} - \left(t - \frac{k}{W}\right)\right]^2 \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{i}{W}\right)^2 - \frac{2}{N} \left(t - \frac{k}{W}\right) \sum_{i=0}^{N-1} \frac{i}{W} + \left(t - \frac{k}{W}\right)^2 \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{i}{W}\right)^2 + \left(t - \frac{k}{W}\right) \left[\left(t - \frac{k}{W}\right) - \frac{N-1}{W}\right]. \end{aligned}$$

The second term above is  $\leq 0$  for  $N \geq 2$  since  $0 \leq t - \frac{k}{W} < \frac{1}{W}$ . Hence

$$\alpha_{W,k}^2(t) \leq \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{i}{W}\right)^2 = \frac{(N-1)(2N-1)}{6W^2} \leq \frac{N^2}{3W^2}.$$

As the bound on  $\alpha_{W,k}^2(t)$  does not depend on  $k$ , we have for all  $t \geq 0$

$$|\text{Bias}[\hat{m}_W(t)]| \leq 2 \omega(m; \frac{N}{\sqrt{3} W}). \quad (14a)$$

Now since  $\{Z_{W,k}\}$  are independent with  $\text{Var}[Z_{W,k}] \leq E[Z_{W,k}^2] = 1$ , it follows from (7) that

$$\text{Var}[\hat{m}_W(t)] = \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Var}[Z_{W,i+k}] \leq \frac{1}{N}. \quad (15a)$$

Thus for all  $t \geq 0$  we have by (14a) and (15a)

$$E[\hat{m}_W(t) - m(t)]^2 \leq 4 \omega^2(m; \frac{N}{\sqrt{3} W}) + \frac{1}{N}$$

and the corresponding bound for  $\hat{s}_W(t)$  follows from (13) and Proposition 1 of the Appendix.

b) For the estimate (8b) and for each fixed  $t \geq 0$  with  $k/W \leq t < (k+1)/W$ , (8b) can be written in the form

$$\hat{m}_W(t) = \sum_{i=k}^{k+N} h_{W,k}(t,i) Z_{W,i} \quad (16)$$

where

$$h_{W,k}(t,i) = \begin{cases} \frac{1}{N} [1 - (Wt - k)] & , i = k \\ \frac{1}{N} & , i = k + 1, \dots, k + N - 1 \\ \frac{1}{N} (Wt - k) & , i = k + N. \end{cases} \quad (17)$$

Then

$$E[\hat{m}_W(t)] = \sum_{i=k}^{k+N} h_{W,k}(t,i) m(\frac{i}{W})$$

which is a positive linear operator on the function  $m(u)$ ,  $u \geq 0$  since  $h_{W,k}(t,i) \geq 0$  and  $\sum_{i=k}^{k+N} h_{W,k}(t,i) = 1$ . It follows, as in Part (a), that

$$|\text{Bias}[\hat{m}_W(t)]| = |E[\hat{m}_W(t)] - m(t)| \leq 2 \omega(m; \alpha_{W,k}(t))$$

where now

$$\begin{aligned} \alpha_{W,k}^2(t) &= \sum_{i=k}^{k+N} \left(\frac{i}{W} - t\right)^2 h_{W,k}(t, i). \\ &= \frac{1}{N} [1 - (Wt - k)] \left(\frac{k}{W} - t\right)^2 + \frac{1}{N} \sum_{i=k+1}^{k+N-1} \left(\frac{i}{W} - t\right)^2 \\ &\quad + \frac{1}{N} (Wt - k) \left(\frac{k+N}{W} - t\right)^2. \end{aligned}$$

After some algebra we obtain

$$\begin{aligned} \alpha_{W,k}^2(t) &= \frac{1}{N} \sum_{j=1}^{N-1} \left[\frac{j}{W} - \left(t - \frac{k}{W}\right)\right]^2 + \frac{(Wt - k)}{NW^2} [N^2 - (2N-1)(Wt - k)] \\ &= \frac{1}{N} \sum_{j=1}^{N-1} \left(\frac{j}{W}\right)^2 + \frac{(Wt - k)}{W^2} [1 - (Wt - k)]. \end{aligned}$$

Summing up the first term above and noting that the second term is bounded by its maximum value  $1/4W^2$  (since  $0 \leq (Wt - k) < 1$ ) we have

$$\begin{aligned} \alpha_{W,k}^2(t) &\leq \frac{(N-1)(2N-1)}{6W^2} + \frac{1}{4W^2} = \frac{1}{3W^2} \left[N^2 - \frac{3}{2}N + \frac{5}{4}\right] \\ &\leq \frac{N^2}{3W^2}. \end{aligned}$$

Thus for all  $t \geq 0$

$$|\text{Bias}[\hat{m}_W(t)]| \leq 2 \omega(m; \frac{N}{\sqrt{3} W}). \quad (14b)$$

From (16) we also have

$$\text{Var}[\hat{m}_W(t)] = \sum_{i=k}^{k+N} h_{W,k}^2(t, i) \text{Var}[Z_{W,i}] \leq \sum_{i=k}^{k+N} h_{W,k}^2(t, i) \quad (18)$$

and by (17)

$$\sum_{i=k}^{k+N} h_{W,k}^2(t,i) = \frac{1}{N^2} \{ [1-(Wt-k)]^2 + (N-1) + (Wt-k)^2 \}.$$

Putting  $x = (Wt-k)$  we have  $0 \leq x < 1$  for which  $\frac{1}{2} \leq (1-x)^2 + x^2 \leq 1$ , so that

$$\frac{1}{N} \left(1 - \frac{1}{2N}\right) \leq \sum_{i=k}^{k+N} h_{W,k}^2(t,i) \leq \frac{1}{N}. \quad (19)$$

It follows that for all  $t \geq 0$ .

$$\text{Var}[\hat{m}_W(t)] \leq \frac{1}{N} \quad (15b)$$

so that by (14b) and (15b) we have for all  $t \geq 0$

$$E[\hat{m}_W(t) - m(t)]^2 \leq 4 \omega^2(m; \frac{N}{\sqrt{3}W}) + \frac{1}{N}$$

and the final result for  $\hat{s}_W(t)$  follows as in Part (a).  $\square$

We now discuss the implications of Theorem 2.1. The first term in the bound is due to the bias and the second is due to the variance of the estimates (8). For a fixed sampling rate  $W$ , the block size  $N$  must be small to reduce the bias but large to reduce the variance. This trade-off is standard in other areas as well (e.g. the window - bandwidth parameter in spectral and probability density estimation). One should therefore use an optimal block size  $N_{\text{opt}}$  which minimizes the bound on the mean-square error. Indeed, when  $s(t)$  is Lip  $\gamma$ ,  $0 < \gamma \leq 1$ , i.e., when  $\omega(s; \delta) = D_s \delta^\gamma$ , we find that  $N_{\text{opt}}$  is given by (the integer part of)

$$N_{\text{opt}} = \left( \frac{3^\gamma K_2}{2^\gamma K_1 D_s^2} \right)^{\frac{1}{1+2\gamma}} \frac{1}{W^{\frac{2\gamma}{1+2\gamma}}} \quad (20)$$

for which the mean-square error becomes

$$E[\hat{s}_W(t) - s(t)]^2 \leq (1 + 2\gamma) \left( \frac{K_1 K_2^{2\gamma} D_s^2}{3^\gamma (2\gamma)^{2\gamma}} \right)^{\frac{2\gamma}{1+2\gamma}} \left( \frac{1}{W} \right)^{\frac{2\gamma}{1+2\gamma}} \quad (21)$$

$$= O(W^{-2\gamma/(1+2\gamma)}).$$

For the (nonsequential) estimates considered in [4], the mean-square error was shown to be  $O(W^{-\min(\gamma, 1/2)})$  and, thus, the estimates of the present paper have faster rates of convergence for all  $0 < \gamma \leq 1$ . For instance, when the signal  $s(t)$  has a bounded derivative  $|s'(t)| \leq D_s$  for all  $t \geq 0$ , then  $\gamma=1$ , the optimal block size  $N_{\text{opt}} = O(W^{2/3})$  and the mean-square error of the present estimates is  $O(W^{-2/3})$  compared to  $O(W^{-1/2})$  for the estimates considered in [4]. It may be of interest to compare these rates to those of other comparable schemes which also convert a continuous-time signal into a binary sequence by periodic sampling and a direct 2-level quantization. One such popular scheme is the standard delta modulation [12] which has been analyzed for stochastic signals only; the most comprehensive analytical study was carried out by Slepian [12] for stationary Gaussian input signals with rational spectral densities, but unfortunately no closed form expressions for the mean-square error and its rate of convergence as  $W \rightarrow \infty$  are available. The only case we are aware of, for which such closed form expressions are available, is that of a Wiener process input [13]. In this case the rate of convergence of the steady-state mean-square error, when an optimal step size is used, is  $W^{-1}$  [13]. For our scheme we have so far improved the rate from  $W^{-1/2}$  to  $W^{-2/3}$ . Note that the sample paths of a Wiener process are almost surely continuous but not differentiable and the comparison to our scheme with Lip 1 signal  $s(t)$  may be somewhat questionable. Still, the rate of convergence  $W^{-1}$  of a standard delta modulator with a Wiener process input provides a performance measure with respect to which the performance of our scheme can be compared. It remains an open question at this point to find the ultimate convergence rate possible for our scheme; we conjecture it to be  $W^{-1}$  (see Section IV) but, so far, we have not found the recovery scheme (receiver) which achieves

this rate.

The actual sampling rate  $W$ , needed to obtain a mean-square error smaller than a given level  $\delta^2$ , can be determined from (21). For example, for signals  $s(t)$  having bounded derivatives  $|s'(t)| \leq D$  for all  $t \geq 0$  we obtain from (20) and (21)

$$W \geq \frac{3K_2\sqrt{K_1} D}{2\delta^3} ; \quad N_{\text{opt}} \geq \frac{3K_2}{2\delta^2} . \quad (22)$$

Note that while the required sampling rate  $W$  is proportional to the variations parameter  $D$  of the signal, the block size  $N$  to be used in the receiver does not depend on the variations of the signal.

When the distribution of the  $\{x_k\}$  in the transmitter is Gaussian or Laplacian, the constants  $K_1$  and  $K_2$  in Theorem 2.1 depend on the parameters  $(\sigma^2, \epsilon)$  under (B2) and on  $(\alpha, \epsilon)$  under (B3). These parameters have so far been left arbitrary positive constants. The question of their optimal values, which minimize the mean-square error, is now discussed when the signal  $s(t)$  is Lip  $\gamma$ ,  $0 < \gamma \leq 1$ . From (21) it is seen that the mean-square error is proportional to  $K_1 K_2^{2\gamma}$ . Minimizing  $K_1 K_2^{2\gamma}$  with respect to  $(\sigma^2, \epsilon)$  under (B2), and with respect to  $(\alpha, \epsilon)$  under (B3), yields

$$\begin{aligned} \sigma &= \left(\frac{1+2\gamma}{2\gamma}\right)^{1/2} (1+y_0)b , & \epsilon &= y_0b; \text{ under (B2)} \\ \alpha^{-1} &= \left(\frac{1+2\gamma}{2\gamma}\right) (1+y_0)b , & \epsilon &= y_0b; \text{ under (B3)} \end{aligned} \quad (23)$$

where  $y_0$  is the positive real root of the cubic equation

$$y^3 = \frac{1}{2\gamma}[y + \gamma(2\gamma+1)].$$

For example, for Lip 1 signals  $s(t)$  the optimal choice is

$$\begin{aligned} \sigma &= 2.8042 b , & \epsilon &= 1.2896 b ; \text{ under (B2)} \\ \alpha^{-1} &= 4.344 b , & \epsilon &= 1.2896 b ; \text{ under (B3)} \end{aligned}$$



It is thus seen that under (B2) or (B3), the variance of the  $\{X_k\}$  in the transmitter should be chosen to be proportional to  $b^2$ , and that the constant  $\epsilon$  in the nonlinearity  $g(x)$  in the receiver should be proportional to  $b$ .

### C. Probability One Convergence and a Central Limit Theorem

We first obtain bounds on the higher order moments of the error  $\hat{s}_W(t) - s(t)$  which provide faster rates of convergence.

Theorem 2.2. Let Assumptions (A) and (B) be satisfied and let the signal  $s(t)$  be Lip  $\gamma$ ,  $0 < \gamma \leq 1$ . If the block size  $N$  is of the form  $N = A W^{2\gamma/(1+2\gamma)}$  (cf. 20), then for every integer  $\ell \geq 1$ , the estimate  $\hat{s}_W(t)$  satisfies

$$E[\hat{s}_W(t) - s(t)]^{2\ell} \leq \frac{K_{\ell,\gamma}(1+o(1))}{W^{2\gamma\ell/(1+2\gamma)}}$$

uniformly in  $t \geq 0$  for some constant  $K_{\ell,\gamma}$ .

Proof. The result for  $\hat{s}_W(t) - s(t)$  follows by Proposition 1 from the result for  $\hat{m}_W(t) - m(t)$  which we now establish. Writing for brevity  $\hat{m}, m$  for  $\hat{m}_W(t), m(t)$  we obtain from  $\hat{m} - m = \text{Bias}[\hat{m}] + (\hat{m} - E[\hat{m}])$

$$E[\hat{m} - m]^{2\ell} = (\text{Bias}[\hat{m}])^{2\ell} + \sum_{j=2}^{2\ell} \binom{2\ell}{j} (\text{Bias}[\hat{m}])^{2\ell-j} E[\hat{m} - E[\hat{m}]]^j. \quad (24)$$

Using the bound on the cumulants of  $\hat{m}$  given in Proposition 2 in the Appendix and the fact that the moments of  $\hat{m}$  can be expressed as finite linear combinations of the cumulants of  $\hat{m}$ , we obtain, as in the proof of Theorem 4.2 of [4], that

$$|E[\hat{m} - m]^j| \leq H_j \frac{1+o(1)}{N^{j-[j/2]}}, \quad j \geq 2 \quad (25)$$

uniformly in  $t \geq 0$  where  $H_j$  is a constant and  $[j/2]$  is the integer part of  $j/2$ . Since  $s(t)$  is Lip  $\gamma$ , so is  $m(t)$  (cf. (13)) and by (14a) - (14b) we have

$$|\text{Bias}[\hat{m}]| \leq 2 \omega(m; \frac{N}{\sqrt{3} W}) = 2D_m \left( \frac{N}{\sqrt{3} W} \right)^\gamma. \quad (26)$$

Putting  $N = A W^{2\gamma/(1+2\gamma)}$  in (25) and (26) and substituting in (24) we obtain

$$E[\hat{m} - m]^{2\ell} \leq (L A^\gamma)^{2\ell} \cdot \frac{1}{W^{2\gamma\ell/(1+2\gamma)}} + (1 + o(1)) \sum_{j=2}^{2\ell} \binom{2\ell}{j} L^{2\ell-j} H_j A^{p_j} \frac{1}{W^{q_j}} \quad (27)$$

where the constants  $L = 2D_m/3^{\gamma/2}$ ,  $p_j = \gamma(2\ell - j) - j + [j/2]$  and

$$q_j = \frac{2\gamma\ell + j\gamma - 2\gamma[j/2]}{1 + 2\gamma}.$$

Since  $q_{2n} = \frac{2\gamma\ell}{1+2\gamma}$ ,  $q_{2n+1} = \frac{2\gamma\ell+\gamma}{1+2\gamma}$ ,  $n = 1, 2, \dots$ ,

it follows that the terms in the sum  $\sum_{j=2}^{2\ell}$  in (27) with  $j$  even are dominant, and thus by (27)

$$E[\hat{m} - m]^{2\ell} \leq \frac{K'_{\ell, \gamma} [1+o(1)]}{W^{2\gamma\ell/(1+2\gamma)}}$$

uniformly in  $t \geq 0$ , where

$$K'_{\ell, \gamma} = (L A^\gamma)^{2\ell} + \sum_{n=1}^{\ell} \binom{2\ell}{2n} L^{2(\ell-n)} A^{p_{2n}} H_{2n}. \quad \square$$

The convergence rate  $O(W^{-2\gamma\ell/(1+2\gamma)})$  given in Theorem 2.2 is again faster than the rate  $O(W^{-\ell \min(\gamma, 1/2)})$  obtained earlier for the nonsequential estimates considered in [4]. Theorem 2.2 implies the convergence with probability one of the estimate  $\hat{s}_W(t)$  to  $s(t)$  as  $W \rightarrow \infty$  (i.e., corresponding to almost every realization of the sequence  $\{X_k\}_{k=0}^{\infty}$  in the transmitter). This strong consistency of  $\hat{s}_W(t)$  together with the rate of convergence is given in the following.

**Theorem 2.3.** Let Assumptions (A) and (B) be satisfied, let the signal  $s(t)$  be Lip  $\gamma$ ,  $0 < \gamma \leq 1$ , and assume the block size  $N$  to be of the form  $N = A W^{2\gamma/(1+2\gamma)}$ . Then for each fixed  $t \geq 0$  we have with probability one

$$(W_0)^\theta \sup_{W \geq W_0} |\hat{s}_W(t) - s(t)| \longrightarrow 0 \text{ as } W_0 \longrightarrow \infty$$

for every constant  $\theta$  satisfying  $0 < \theta < \frac{\gamma}{1+2\gamma}$ .

Proof. We note that for each fixed  $t \geq 0$ , the estimate  $\hat{s}_W(t)$ , regarded as a random process with parameter  $W > 0$ , is separable. The result then follows from Theorem 2.2 and Kolmogorov's theorem (Neveu [14, p. 97]) in the manner of the proof of Theorem 4.3 in [4].  $\square$

For example, when  $s(t)$  has a bounded derivative then  $\gamma = 1$  and with probability one we have, in particular,

$$W^\theta |\hat{s}_W(t) - s(t)| \longrightarrow 0 \text{ as } W \longrightarrow \infty$$

for all  $0 < \theta < 1/3$ . for the estimates considered in [4] we obtained  $0 < \theta < 1/4$  for the same example and, thus, the sequential estimates of the present paper have a faster rate of almost sure convergence.

We finally derive a central limit theorem for the estimation error  $\hat{s}_W(t) - s(t)$  which is useful in obtaining confidence intervals for this error. Define the normalized error process

$$\tilde{s}_W(t) = B_W(t)[\hat{s}_W(t) - s(t)], \quad t \geq 0$$

where

$$B_W(t) = u'[s(t)] \text{Var}^{-1/2}[\hat{m}_W(t)]. \quad (28)$$

In the following we shall assume that under (B1),  $X$  is uniform over  $[-c, c]$  with  $c > b$  in which case the estimate (10) is replaced by  $\hat{s}_W(t) = c \hat{m}_W(t)$ .

Theorem 2.4. Let Assumptions (A) and (B) be satisfied and let the signal  $s(t)$  be Lip  $\gamma$ ,  $0 < \gamma \leq 1$ . Assume, in addition, that under (B1),  $X$  is uniform over  $[-c, c]$  with  $c > b$ . If the block size  $N$  is chosen to be of the form

$$N = A W^\lambda, \quad 0 < \lambda < \frac{2\gamma}{1+2\gamma},$$

then for each fixed  $t \geq 0$ ,  $\xi_W(t)$  is asymptotically standard normal variable as  $W \rightarrow \infty$  and for distinct  $t$ 's the values of the process  $\{\xi_W(t), t \geq 0\}$  are asymptotically independent.

The normalizing factor  $B_W(t)$  is bounded from above and below, uniformly in  $t \geq 0$ , as follows:

$$\text{For (8a): } M_1 N^{1/2} \leq B_W(t) \leq M_2 N^{1/2} \quad (29)$$

$$\text{For (8b): } M_1 N^{1/2} \leq B_W(t) \leq M_2 \frac{N^{1/2}}{\sqrt{1-(1/2N)}}$$

where

$$\text{for (B1): } M_1 = 1/c \quad ; \quad M_2 = 1/\sqrt{c^2 - b^2},$$

$$\text{for (B2): } M_1 = \sqrt{2/\pi\sigma^2} \exp(-b^2/2\sigma^2) \quad ; \quad M_2 = (2\pi\sigma^2 \phi(b/\sigma) [1-\phi(b/\sigma)])^{-1/2},$$

$$\text{for (B3): } M_1 = \alpha \exp(-\alpha b) \quad ; \quad M_2 = \alpha \exp(\alpha b/2) [2-\exp(-\alpha b)]^{-1/2}.$$

Proof. We prove the asymptotic normality and independence of the normalized error process for  $m(t)$ :

$$\tilde{m}_W(t) = \frac{\hat{m}_W(t) - m(t)}{\text{Var}^{1/2}[\hat{m}_W(t)]}, \quad t \geq 0.$$

The corresponding results for the signal  $s(t)$ , as stated in the theorem, will then follow by using a result of Mann and Wald [15, p.226] in the manner of the proof of Theorem 3.4 in [4]. Putting

$$\xi_W(t) = \frac{\hat{m}_W(t) - E[\hat{m}_W(t)]}{\text{Var}^{1/2}[\hat{m}_W(t)]} \quad t \geq 0$$

we have

$$\tilde{m}_W(t) = \varepsilon_W(t) + \frac{\text{Bias}[\hat{m}_W(t)]}{\text{Var}^{1/2}[\hat{m}_W(t)]}, \quad t \geq 0.$$

The proof is accomplished by showing that as  $W \rightarrow \infty$  the second term tends to zero and  $\varepsilon_W(t)$  is asymptotically standard normal variable with asymptotically independent values for distinct  $t$ 's.

We have  $V_2 \leq \text{Var}[Z_{W,i}] \leq 1$  where

$$\begin{aligned} V_2 &= \min_{|s| \leq b} \text{Var}[\text{sgn}(s+X)] = 1 - \max_{|s| \leq b} u^2(s) \\ &= 1 - u^2(b). \end{aligned}$$

$V_2$  is easily calculated under (B2), (B3) and the modified (B1) and in all cases  $V_2 > 0$ . Thus by (15a) for the estimate (8a) and (18) and (19) for the estimate (8b) we have

$$\text{for (8a):} \quad \frac{V_2}{N} \leq \text{Var}[\hat{m}_W(t)] \leq \frac{1}{N} \quad (30a)$$

$$\text{for (8b):} \quad \frac{V_2}{N} \left(1 - \frac{1}{2N}\right) \leq \text{Var}[\hat{m}_W(t)] \leq \frac{1}{N}. \quad (30b)$$

Hence by (14a) and the lower bound in (30a) we have for the estimate (8a)

$$\frac{|\text{Bias}[\hat{m}_W(t)]|}{\text{Var}^{1/2}[\hat{m}_W(t)]} = \left( \frac{N^Y + 1/2}{W^Y} \right) \rightarrow 0 \text{ as } W \rightarrow \infty$$

by assumption on  $N$ . Similarly for the estimate (8b) (cf. (14b) and (30b)).

We now consider  $\varepsilon_W(t)$ . Clearly  $E[\varepsilon_W(t)] = 0$  and  $\text{Var}[\varepsilon_W(t)] = 1$  for all  $t \geq 0$ . Also for  $r \geq 3$  and all instants  $t_1, \dots, t_r \geq 0$  (not necessarily distinct) we have

$$\text{Cum}_r\{\varepsilon_W(t_1), \dots, \varepsilon_W(t_r)\} = \frac{\text{Cum}_r\{\hat{m}_W(t_1), \dots, \hat{m}_W(t_r)\}}{\prod_{i=1}^r \text{Var}^{1/2}[\hat{m}_W(t_i)]}$$

and using Proposition 2 for the numerator and the lower bound in (30a)-(30b) for the denominator we find

$$\text{Cum}_r\{\varepsilon_W(t_1), \dots, \varepsilon_W(t_r)\} = o(N^{-(\frac{r}{2} - 1)}) \rightarrow 0 \text{ as } W \rightarrow \infty$$

since  $r \geq 3$ . Finally, for  $t_1 \neq t_2$  we have  $E[\varepsilon_W(t_1)\varepsilon_W(t_2)] \rightarrow 0$  as  $W \rightarrow \infty$  since for large enough  $W$  (say  $\frac{2N}{W} < |t_1 - t_2|$ ),  $\varepsilon_W(t_1)$  and  $\varepsilon_W(t_2)$  are independent as  $\hat{m}_W(t_1)$  and  $\hat{m}_W(t_2)$  are expressed in terms of nonoverlapping blocks of  $N Z_{W,i}$ 's. It then follows by Lemma P4.5 of [16] that the finite dimensional distributions of  $\{\varepsilon_W(t), t \geq 0\}$  converge as  $W \rightarrow \infty$  to the finite dimensional distributions of a Gaussian process with mean zero and covariance  $R(t_1, t_2) = 1$  for  $t_1 = t_2$  and  $R(t_1, t_2) = 0$  for  $t_1 \neq t_2$ ; which establishes the desired result for  $\hat{m}_W(t)$ .

The bounds (29) for the normalization factor  $B_W(t)$  follow from (28), (30) and the observation that

$$\mu'(b) \leq \mu'[s(t)] \leq \mu'(0). \quad \square$$

### III. TRANSMISSION OVER A NOISY CHANNEL - ERROR ANALYSIS

In this section we study the degradation in the performance of the receiver of Section II when the binary data  $\{Z_{W,k}\}$  is transmitted over a noisy channel.

The modification in the transmitter/receiver structure of Figure 1 is shown in Figure 2. The binary sequence  $\{Z_{W,k}\}$  is now pulse modulated

$$p(t) = \sum_{k=0}^{\infty} Z_{W,k} a(t - \frac{k}{W}), \quad t \geq 0$$

where  $a(t)$  is the transmission filter, i.e.,  $a(t)$  is a fixed function over  $[0, 1/W]$ , vanishing outside of  $[0, 1/W]$ , with finite energy

$$e_W = \int_0^{1/W} |a(t)|^2 dt.$$

The power of the transmitter is then

$$P_W = W e_W = W \int_0^{1/W} |a(t)|^2 dt. \quad (31)$$

A possible choice for  $a(t)$  is, for example,  $a(t) = C \sin \pi W t \mathbf{1}_{[0, 1/W]}(t)$  for which  $p(t)$  resembles a PSK waveform and the transmitter power is then  $P_W = C^2/2$ . The channel noise  $\{\eta(t), -\infty < t < \infty\}$  is assumed to be a wide-sense stationary process, independent of the sequence  $\{X_k\}$ , with mean zero and covariance function  $R(t)$ . The channel noise need not be Gaussian nor white. The received waveform is then

$$r(t) = p(t) + \eta(t), \quad t \geq 0.$$

The modification in the receiver, as shown in Figure 2, is based on the simple idea of first estimating  $Z_{W,k}$  from the received waveform  $r(t)$  over the  $k^{\text{th}}$  interval  $[\frac{k}{W}, \frac{k+1}{W}]$ , and then using the estimates  $\{\hat{Z}_{W,k}\}$  as the input to our previous (noiseless channel) receiver. This is accomplished by employing a standard matched filter whose output

$$T_k = \int_{k/W}^{(k+1)/W} a(t - \frac{k}{W}) r(t) dt, \quad k=0,1,\dots, \quad (32)$$

is used to estimate  $Z_{W,k}$  by

$$\hat{Z}_{W,k} = \psi(T_k), \quad k = 0,1,\dots \quad (33)$$

where the transformation  $\psi(x)$  is specified below. Hence the estimate  $\hat{s}_W(t)$  of  $s(t)$  is given by (9)

$$\hat{s}_W(t) = g[\hat{m}_W(t)], \quad t \geq 0 \quad (9)$$

where, as before,  $g(x)$  is specified in Assumption B but with  $\hat{m}_W(t)$  is now determined from

$$\hat{m}_W(\frac{k}{W}) = \frac{1}{N} \sum_{i=0}^{N-1} \hat{Z}_{W,k+i}, \quad k = 0,1,\dots \quad (7')$$

by (8a) or (8b). For convenience of notation we shall write  $\hat{m}_W(t)$  in the form

$$\hat{m}_W(t) = \sum_{i=0}^{\infty} h_W(t,i) \hat{z}_{W,i}, \quad t \geq 0 \quad (8')$$

where the kernel  $\{h_W(t,i)\}$  can easily be identified for the sequential estimates (8a)-(8b). In fact, the analysis in this section is valid for any  $\{h_W(t,i)\}$  corresponding to a positive linear operator with

- i.  $h_W(t,i) \geq 0$  for all  $t \geq 0, i \geq 0$ ,
  - ii.  $\sum_{i=0}^{\infty} h_W(t,i) = 1$  for all  $t \geq 0$ ,
  - iii.  $\sum_{i=0}^{\infty} i^2 h_W(t,i) < \infty$  for all  $t \geq 0$ ,
  - iv.  $\sum_{i=0}^{\infty} h_W^2(t,i) < \infty$  for all  $t \geq 0$ .
- (34)

Consider now the question of choosing the nonlinearity  $\psi(x)$  in (33). It is natural to base the choice on minimizing the mean-square error  $E[\hat{z}_{W,k} - z_{W,k}]^2$  under the constraint that  $\hat{z}_{W,k}$  takes on the values  $\pm 1$ . For example, when the channel noise is white and Gaussian with  $R(t) = (\nu_0/2)\delta(t)$ , the analysis gives  $\psi(x) = \text{sgn } x$  (i.e.,  $\hat{z}_{W,k} = \text{sgn}[T_k]$ ) which corresponds to the classical optimal detector (nonparametric in the signal  $s(t)$ )-see Part C of Appendix -. For this case one finds  $E[\hat{z}_{W,k} - z_{W,k}]^2 = 4[1 - \Phi(d_W)]$  where  $d_W = \sqrt{2e_W/\nu_0}$  is "the signal to channel-noise ratio". However, as seen from the Appendix, this optimal detector for the symbol  $z_{W,k}$  is not necessarily optimal as far as the estimation of  $s(t)$  is concerned. In fact, the linear choice  $\psi(x) = x/e_W$ , which gives the worst possible error  $E[\hat{z}_{W,k} - z_{W,k}]^2 = 1$  under the condition  $d_W=1$ , provides a smaller degradation in the estimation of  $s(t)$  than the "optimal"  $\psi(x) = \text{sgn } x$ . Consequently, we shall assume in this section that

$$\hat{z}_{W,k} = \frac{T_k}{e_W}, \quad k = 0, 1, \dots \quad (35)$$



which is simply a scaling of the matched filter output.

We have,

Theorem 3.1. Let Assumptions (A) and (B) be satisfied and the estimate  $\hat{s}_W(t)$  be given by (9) and (8') with  $\hat{z}_{W,i}$  given by (35) and  $\{h_W(t,i)\}$  satisfying (34).

Then

a) When the channel noise is white, with  $R(t) = (v_0/2)\delta(t)$ , we have for each fixed  $t \geq 0$ ,

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \omega^2(s; \alpha_W(t)) + K_2 \left(1 + \frac{1}{d_W^2}\right) v_W^2(t).$$

b) When the channel noise is colored, with arbitrary continuous correlation  $R(t)$ , we have for each fixed  $t \geq 0$ ,

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \omega^2(s; \alpha_W(t)) + K_2 (v_W^2(t) + \frac{R(0)}{We_W}),$$

where in both parts (a) and (b)

$$\alpha_W^2(t) = \sum_{i=0}^{\infty} (t - \frac{i}{W})^2 h_W(t,i),$$

$$v_W^2(t) = \sum_{i=0}^{\infty} h_W^2(t,i),$$

$$d_W = \sqrt{2e_W/v_0},$$

and the constants  $K_1$  and  $K_2$  are as in Theorem 2.1.

Corollary 3.1. Assume  $\{h_W(t,i)\}$  corresponds to the sliding windows of Section II (cf. (8a)-(8b)), then

a) When the channel noise is white, the sequential estimates  $\hat{s}_W(t)$  satisfy, uniformly in  $t \geq 0$ ,

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \omega^2(s; \frac{N}{\sqrt{3}W}) + \frac{K_2}{N} (1 + \frac{1}{d_W^2})$$

b) When the channel noise is colored, the sequential estimates  $\hat{s}_W(t)$  satisfy, uniformly in  $t \geq 0$ ,

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \omega^2(s; \frac{N}{\sqrt{3}W}) + K_2 (\frac{1}{N} + \frac{R(0)}{We_W}).$$

A comparison of Corollary 3.1 with Theorem 2.1 shows that the channel noise simply increases the variance of the estimate  $\hat{s}_W(t)$  by a factor inversely proportional to the "signal to channel-noise power ratio" (e.g.  $We_W/R(0)$  in the colored noise case).

Proof. By (32) we have

$$T_k = Z_{W,k} \int_0^{1/W} |a(t)|^2 dt + \int_0^{1/W} a(t) \eta(t - \frac{k}{W}) dt$$

so that by (35)

$$\hat{Z}_{W,k} = Z_{W,k} + \zeta_k, \quad k = 0, 1, \dots \quad (36)$$

where  $\{\zeta_k\}$  is a wide-sense stationary sequence, independent of  $\{Z_{W,k}\}$ , with mean zero and covariance sequence  $\rho_n = E[\zeta_{n+k} \zeta_k^*]$  given by

$$\rho_n = \begin{cases} \frac{1}{2e_W} \iint_0^{1/W} a(t)a(\tau)R(t-\tau + \frac{n}{W})dt d\tau, & \text{noise is colored} \\ \frac{1}{2e_W} \delta_{n,0} & , \text{noise is white} \end{cases} \quad (37)$$

Since the  $Z_{W,k}$ 's are independent with mean  $m(k/W)$  and  $\text{Var}[Z_{W,k}] \leq 1$ , we have by (36)

$$E[\hat{Z}_{W,k}] = m(k/W)$$

and

$$|\text{Cov}\{\hat{Z}_{W,i}, \hat{Z}_{W,j}\}| \leq \begin{cases} \delta_{i,j} + |\rho_{i-j}| & , \text{ noise is colored} \\ [1 + \frac{\nu_0}{2e_W}] \delta_{i,j} & , \text{ noise is white.} \end{cases} \quad (38)$$

Hence for the estimate  $\hat{m}_W(t)$ , given by (8'), we have

$$E[\hat{m}_W(t)] = \sum_{i=0}^{\infty} h_W(t,i) m(i/W) \quad (39)$$

and

$$\text{Var}[\hat{m}_W(t)] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}\{\hat{Z}_{W,i}, \hat{Z}_{W,j}\} h_W(t,i) h_W(t,j). \quad (40)$$

Since  $m(t)$ ,  $t \geq 0$ , is a uniformly continuous function, we have by the result of [11], on the interpolation of continuous functions by positive linear operators, and (39) that for each fixed  $t \geq 0$

$$|\text{Bias}[\hat{m}_W(t)]| \leq 2 \omega(m; \alpha_W(t)) \quad (41)$$

where  $\alpha_W^2(t)$  is given in the theorem. For the variance expression (40) we consider the white noise case first. Then by (38)

$$\text{Var}[\hat{m}_W(t)] \leq (1 + \frac{1}{d_W^2}) \sum_{i=0}^{\infty} h_W^2(t,i) = (1 + \frac{1}{d_W^2}) v_W^2(t). \quad (42a)$$

When the channel noise is colored we have  $|\rho_{i-j}| \leq \rho_0$ , and by (37)

$$\rho_0 \leq \frac{R(0)}{e_W^2} \left[ \int_0^{1/W} |a(t)| dt \right]^2 \leq \frac{R(0)}{We_W}$$

where we have used the Cauchy-Schwarz inequality in the last step. Hence by (38) and (40)

$$\begin{aligned} \text{Var}[\hat{m}_W(t)] &\leq \sum_{i=0}^{\infty} h_W^2(t,i) + \frac{R(0)}{We_W} \left[ \sum_{i=0}^{\infty} h_W(t,i) \right]^2 \\ &= v_W^2(t) + \frac{R(0)}{We_W} \end{aligned} \quad (42b)$$

where the last step follows by (34.ii). We thus have by (41) and (42),

$$E[\hat{m}_W(t) - m(t)]^2 \leq \begin{cases} 4\omega^2(m; \alpha_W(t)) + (1 + \frac{1}{d_W^2})v_W^2(t), & \text{noise is white} \\ 4\omega^2(m; \alpha_W(t)) + v_W^2(t) + \frac{R(0)}{We_W}, & \text{noise is colored} \end{cases}$$

and the result for  $\hat{s}_W(t)$  follows from (13) and Proposition 1 in the Appendix.  $\square$

The implications of Corollary 3.1 are now considered in more details. In the case of colored noise, the contribution of the channel noise to the mean-square error of  $\hat{s}_W(t)$  is the additive term  $K_2 R(0)/We_W = K_2 R(0)/P_W$ , where  $P_W$  is the power of the transmitter. Thus to combat the channel noise,  $P_W$  must be proportional to  $W^\lambda$  for some  $\lambda > 0$ . In the case of white noise, the contribution is the additive term  $K_2/Nd_W^2 = K_2 v_0(W/N)/P_W$ . If the block size  $N$  is chosen to be  $O(W^{2\gamma/(1+2\gamma)})$  for Lip  $\gamma$  signals (cf.  $N_{\text{opt}}$  given in (20)), then  $P_W$  must be proportional to  $W^{[2\gamma/(1+2\gamma)] + \lambda}$  for some  $\lambda > 0$  in order to combat the channel noise. It is then clear that the transmission power  $P_W$  must be appreciably higher in the white noise case than in the colored noise case for the same channel noise contribution to the mean-square error of  $\hat{s}_W(t)$ . It is also of practical interest to obtain the values of the parameters  $(W, N, P)$  for the reconstruction of the signal  $s(t)$  to be achieved with mean-square error not exceeding a given level  $\delta^2$ . For simplicity we carry out the analysis for signals  $s(t)$  having bounded derivatives  $|s'(t)| \leq D$  (i.e.,  $s(t) \in \text{Lip } 1$ ) and a sinusoidal pulse

$$a(t) = \begin{cases} C \sin \pi W t, & 0 \leq t \leq \frac{1}{W} \\ 0, & \text{elsewhere} \end{cases} \quad (43)$$

is used as a transmission filter.

a) White Channel Noise. Choose  $C^2 = \nu_0 W$ . Then  $d_W^2 = 1$  and the channel noise simply doubles the variance of the estimate  $\hat{s}_W(t)$ . We have by Corollary 3.1

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 D^2 \left( \frac{N}{\sqrt{3} W} \right)^2 + \frac{2K_2}{N}.$$

Minimizing the right hand side with respect to  $N$ , we obtain the optimal block size to be (the integer part of)

$$N_{\text{opt}} = \left\{ \frac{3K_2}{K_1 D^2} \right\}^{1/3} W^{2/3}$$

for which

$$E[\hat{s}_W(t) - s(t)]^2 \leq 3 \left( \frac{K_1 K_2^2 D^2}{3} \right)^{1/3} \left( \frac{1}{W} \right)^{2/3}.$$

Hence, for the mean-square error to be less than  $\delta^2$ , we require the values of  $(W, N, C^2)$  to be

$$W \geq 3\sqrt{K_1} K_2 (D/\delta^3), \quad N = 3K_2 (1/\delta^2), \quad C^2 = 3 \nu_0 \sqrt{K_1} K_2 (D/\delta^3). \quad (44)$$

Under (B1) - the simplest transmitter/receiver structure - these values are

$$W \geq 6b^2 (D/\delta^3), \quad N = 3b^2 (1/\delta^2), \quad C^2 = 6 \nu_0 b^2 (D/\delta^3). \quad (45)$$

b) Colored Channel Noise. By Corollary 3.1

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 D^2 \left( \frac{N}{\sqrt{3} W} \right)^2 + K_2 \left( \frac{1}{N} + \frac{2R(0)}{C^2} \right).$$

Minimizing the right hand side relative to  $N$ , we find

$$N_{\text{opt}} = \left\{ \frac{3K_2}{2K_1 D^2} \right\}^{1/3} W^{2/3}$$

and if we demand, as in case (a), that the noise contribution simply doubles the variance we find

$$C^2 = 2R(0)N_{\text{opt}}$$

and for the mean-square error to be less than  $\delta^2$  we need the values of  $(W, N, C^2)$  to be

$$W \geq \sqrt{125/12} \sqrt{K_1} K_2 (D/\delta^3), \quad N = \frac{5}{2} K_2 (1/\delta^2), \quad C^2 = 5R(0) K_2 (1/\delta^2).$$

(46)

Comparing (44) with (46) we note that the transmitter power  $C^2$  in the white noise case is proportional to the variation parameter  $D$  of the signal and to  $(1/\delta^3)$  whereas in the colored noise case the transmitter power is independent of  $D$  and proportional to  $1/\delta^2$  only.

#### IV. COMMENTS

We point out some open problems connected with the reconstruction scheme considered in this paper.

We first note that the results of this paper generalize and sharpen those of [4]. In particular, the mean-square convergence rate obtained here is  $W^{-2/3}$  compared to  $W^{-1/2}$  for the nonsequential estimates of [4]. An open problem is therefore to find the ultimate mean-square convergence rate of any recovery scheme (sequential or not) based on the binary data  $\{Z_{W,k}\}_k$ . We believe this rate to be  $W^{-1}$  for nonconstant signals  $s(t)$  (when  $s(t)$  is constant for all  $t$ , the problem is trivial). One reason for this belief is the nature of the trade-off between bias and variance in Theorem 2.1 (as a function of the block size  $N$ ) which is reminiscent of a similar trade-off in spectral and probability density estimation (as a function of the window-bandwidth parameter). This problem is

currently under investigation.

A second open problem is to extend the results of this paper to the case where the signal  $s(t)$  is not necessarily uniformly bounded. It is clear that in such a case the contaminating sequence  $\{X_k\}$  should have a strictly monotonic distribution  $F(x)$  over  $(-\infty, \infty)$  (e.g., Gaussian). Then  $\mu(s)$  is strictly monotonic on  $(-\infty, \infty)$ ,  $s(t) = \mu^{-1}[m(t)]$ , and we set  $\hat{s}(t) = \mu^{-1}[\hat{m}(t)]$ . Now it is possible to show that  $\hat{m}(t)$  converges to  $m(t)$  with probability one and thus, also,  $\hat{s}(t)$  to  $s(t)$  since  $\mu^{-1}(x)$  is a continuous function. The main problem in this case is to obtain bounds on the mean-square error for  $\hat{s}(t)$ ; the difficulty being that such bounds cannot be obtained from those for  $\hat{m}(t)$ , as in the proof of Theorem 2.1, since  $\mu^{-1}(x)$  is not Lip 1 on  $[-1, 1]$ .

The question of extending the results of this paper to stochastic signals is currently under investigation.

## APPENDIX

Collected here are two propositions needed in the proofs of the theorems in Section II as well as a supplement to the white channel noise case of Section III.

The first proposition provides the link between the properties of  $\hat{s}_W(t)$  and  $\hat{m}_W(t)$ .

A. Proposition 1. Under Assumptions (A) and (B) we have: Under (B1)

$$\hat{s}_W(t) - s(t) = b[\hat{m}_W(t) - m(t)]$$

and under (B2) or (B3) we have for each integer  $p \geq 1$ .

$$E|\hat{s}_W(t) - s(t)|^p \leq \Lambda_p E|\hat{m}_W(t) - m(t)|^p$$

where

$$\text{for (B2): } \Lambda_p = (\pi/2)^{p/2} \sigma^p e^{pc^2/2\sigma^2} [1 + (b/\epsilon)^p]$$

$$\text{for (B3): } \Lambda_p = \alpha^{-p} e^{p\alpha c} [1 + (b/\epsilon)^p].$$

Proof. See Proposition 5.1 in [4].  $\square$

The second proposition provides upper bounds on the cumulants of the estimate  $\hat{m}_W(t)$ .

B. Proposition 2. For every integer  $r \geq 2$  and every choice of instants  $t_1, \dots, t_r \geq 0$ , the joint cumulant of order  $r$  of the estimates  $\hat{m}_W(t)$  (cf. (8a)-(8b)) satisfies

$$|\text{Cum}_r\{\hat{m}_W(t_1), \dots, \hat{m}_W(t_r)\}| \leq \frac{\Gamma_r}{N^{r-1}}$$

uniformly in  $\{t_j\}$  for some finite constant  $\Gamma_r$ .

Proof. Assume without loss of generality that  $t_j \in [\frac{k_j}{W}, \frac{k_j+1}{W})$ ,  $j = 1, \dots, r$ , where the integers  $k_1, \dots, k_r$  are not necessarily distinct. Then we can write

$$\hat{m}_W(t_j) = \sum_{i=k_j}^{k_j+N} h_{W,k_j}(t_j, i) Z_{W,i}$$

where for the estimate (8a)

$$h_{W,k_j}(t_j, i) = \begin{cases} \frac{1}{N} & , i = 0, 1, \dots, N-1 \\ 0 & , \text{otherwise} \end{cases} \quad (A1)$$

and for the estimate (8b) they are given by (17). By Proposition 4.2 of [4] we have

$$|\text{Cum}_r\{\hat{m}_W(t_1), \dots, \hat{m}_W(t_r)\}| \leq \Gamma_r \sum_{i \in I} \prod_{j=1}^r h_{W,k_j}(t_j, i) \quad (A2)$$

where

$$I = \cap_{j=1}^r I_j \quad ; \quad I_j = \{k_j, \dots, k_j + N\}.$$



For the estimate (8a) we have by (A1)

$$\sum_{i \in I} \prod_{j=1}^r h_{W,k_j}(t_j, i) = \sum_{i \in I'} \frac{1}{N^r}$$

$$\leq N \cdot \frac{1}{N^r} = \frac{1}{N^{r-1}}$$

where  $I' = \bigcap_{j=1}^r \{k_j, \dots, k_j + N-1\}$  and the inequality follows from the cardinality of  $I'$  being at most  $N$ . for the estimate (8b) we have from (A2) and the  $r^{\text{th}}$  dimensional version of Hölders inequality

$$|\text{Cum}_r(\hat{m}_W(t_1), \dots, \hat{m}_W(t_r))| \leq r_r \prod_{j=1}^r \left\{ \sum_{i \in I} [h_{W,k_j}(t_j, i)]^r \right\}^{1/r}. \quad (\text{A3})$$

Now by (17)

$$\sum_{i \in I} [h_{W,k_j}(t_j, i)]^r \leq \sum_{i \in I_j} [h_{W,k_j}(t_j, i)]^r$$

$$= \frac{1}{N^r} \{ [1 - (Wt_j - k_j)]^r + (N - 1) + (Wt_j - k_j)^r \}$$

$$\leq \frac{1}{N^{r-1}} \quad (\text{A4})$$

where the last step follows from  $(1-x)^r + x^r \leq 1$  for  $0 \leq x \leq 1$ . The result now follows by substituting (A4) in (A3).  $\square$

C. Supplement to Section III. Assume the channel noise  $\{n(t), -\infty < t < \infty\}$  to be white and Gaussian with  $R(t) = (v_0/2) \delta(t)$ . Let

$$\hat{Z}_{W,k} = \text{sgn}(T_k - \theta), \quad k=0,1,\dots$$

for some threshold  $\theta$  which minimizes  $E[\hat{Z}_{W,k} - Z_{W,k}]^2$ . It is not difficult to see that

$$E[\hat{Z}_{W,k} - Z_{W,k}]^2 = 4Pe$$

where  $P_e$  is the probability of error per symbol given by

$$P_e = \Phi\left(\frac{\theta - e_W}{\sqrt{v_0 e_W/2}}\right) \left[\frac{1 + m(k/W)}{2}\right] + \left[1 - \Phi\left(\frac{\theta + e_W}{\sqrt{v_0 e_W/2}}\right)\right] \left[\frac{1 - m(k/W)}{2}\right].$$

the value of  $\theta$  which minimizes  $P_e$  is

$$\theta = \frac{1}{2d_W} \ln \frac{1 - m(k/W)}{1 + m(k/W)}$$

which is dependent on  $m(k/W) = \mu[s(k/W)]$ . Since  $s(t)$  is unknown, a nonparametric choice of  $\theta$  is  $\theta = 0$  for which

$$\hat{Z}_{W,k} = \text{sgn}[T_k] \quad (A5)$$

and

$$E[\hat{Z}_{W,k} - Z_{W,k}]^2 = 4[1 - \Phi(d_W)] \quad (A6)$$

where

$$d_W = \sqrt{2e_W/v_0} \quad (A7)$$

We then have

Proposition C. Under the assumptions of Theorem 3.1, but with  $\hat{Z}_{W,k}$  given by (A5), we have when the channel noise is white and Gaussian

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \{\omega(s; \alpha_W(t)) + \frac{1}{Q}[1 - \Phi(d_W)]\}^2 + K_2 v_W^2(t)$$

where  $\alpha_W^2(t)$ ,  $v_W^2(t)$ , and the constants  $K_1$  and  $K_2$  are as in Theorem 3.1 and the constant  $Q$  is given by (12).

Corollary C. Under the assumptions of Corollary 3.1, we have, uniformly in  $t \geq 0$ ,

$$E[\hat{s}_W(t) - s(t)]^2 \leq K_1 \left\{ \omega\left(s; \frac{N}{\sqrt{3}W}\right) + \frac{1}{Q}[1 - \Phi(d_W)] \right\}^2 + \frac{K_2}{N}$$

Proof. The derivations proceed in the manner of the proof of Theorem 3.1 noting that for  $\hat{Z}_{W,k}$ , given by (A5), we now have

$$E[\hat{Z}_{W,i}] = [2\phi(d_W) - 1] m(i/W)$$

$$|\text{Cov}\{\hat{Z}_{W,i}, \hat{Z}_{W,j}\}| \leq E[\hat{Z}_{W,i}]^2 \delta_{i,j} = \delta_{i,j}.$$

Then

$$\begin{aligned} |\text{Bias}[\hat{m}_W(t)]| &\leq 2[2\phi(d_W) - 1] \omega(m; \alpha_W(t)) + 2[1 - \phi(d_W)] |m(t)| \\ &\leq 2\{\omega(m; \alpha_W(t)) + [1 - \phi(d_W)]\} \end{aligned}$$

and

$$\text{Var}[\hat{m}_W(t)] \leq \sum_{i=0}^{\infty} h_W^2(t, i) = v_W^2(t)$$

and the result follows.  $\square$

Note that the channel noise here increases the bias of the estimate  $\hat{s}_W(t)$  in contrast to Theorem 3.1 (a). Corollary C implies that the additional term, due to the channel noise, becomes negligible only when the "signal to channel-noise ratio"  $d_W$  tends to infinity as  $W \rightarrow \infty$ . In sharp contrast, Corollary 3.1 (a) implies that the additional term  $K_2/d_W^2 N$ , due to the channel noise, becomes negligible as  $W \rightarrow \infty$  (and thus  $N \rightarrow \infty$ ) even if  $d_W = 1$ , say. Thus, with  $d_W = 1$ , the noise contribution when  $\hat{Z}_{W,k}$  is nonoptimal (35) will be smaller, for large  $W$ , than when  $\hat{Z}_{W,k}$  is optimal (A5). This conclusion holds even though with  $d_W = 1$  we find for the nonoptimal estimate (35) that (cf. (36), (37))

$$E[\hat{Z}_{W,k} - Z_{W,k}]^2 = 1.$$

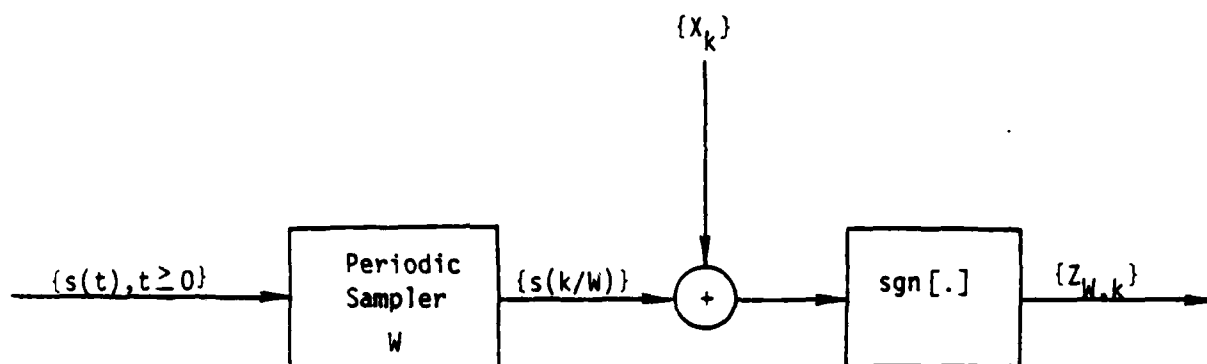
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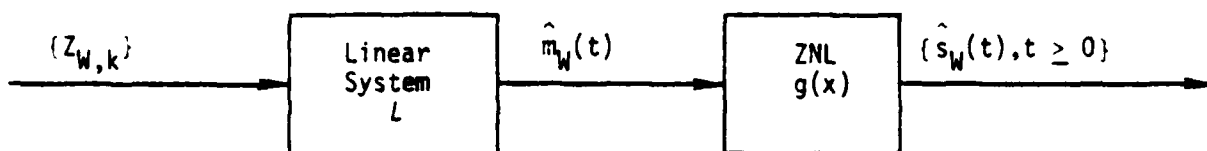
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Transmitter



Receiver

Fig. 1 The structure of the transmitter/receiver model - noiseless channel

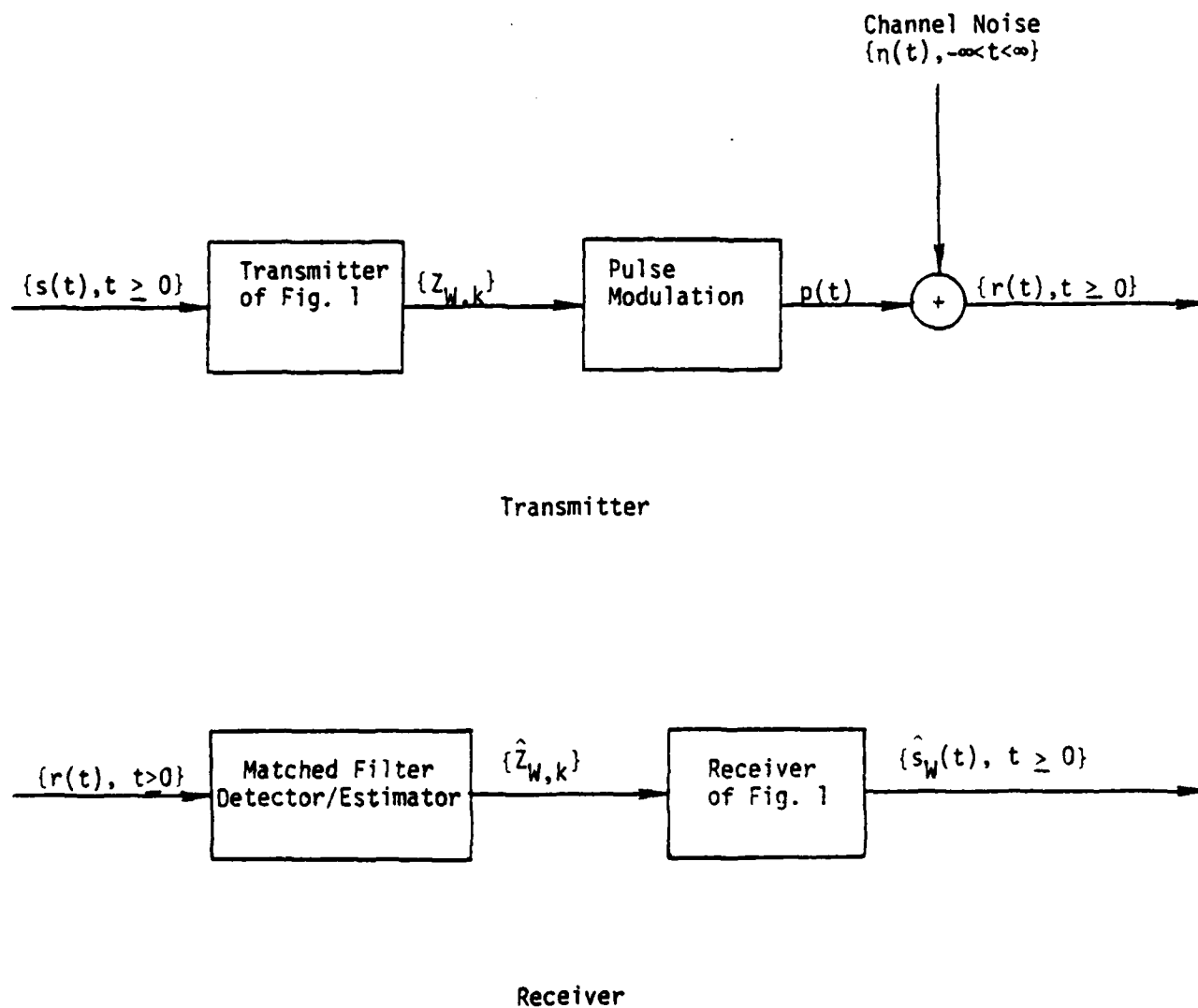


Fig. 2 Modification of the transmitter/receiver in the presence of channel noise.

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The reconstruction of continuous-time signals $s(t)$ from the sign of their (deliberately) contaminated samples is considered. Sequential, generally nonlinear estimates of $s(t)$ are established and their performance is studied; error bounds and convergence rates are derived. The signal $s(t)$ need not be bandlimited. The convergence rates obtained here are faster than those obtained in [4] for nonsequential estimates. The degradation in the reconstruction of the signal, due to transmission over an arbitrary noisy channel, is also investigated and bounds on the additional error are obtained.		

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